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Differential Transformation Method for Solving Nonlinear Differential Equations*

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Abstract: The differential transformation method for solving nonlinear differential equations is investigated in this paper. This method is a feasible tool for obtaining the exact solution of differetial equations. The validity of the method is demonstrated by some test problems. The results show the reliability and efficiency of the method.

Keywords: nonlinear differential equation; differential transformation method; initial value problem

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1 Introduction

Differential transformation method (DTM) was first introduced by Zhou[1], who solved linear and nonlinear problems in electrical circuit problems. The method and related theorems are well addressed in [2,3].

In the paper, we consider the nonlinear differential equation

$$u_t = u_{xx} + \lambda_1 u u_x + \lambda_2 u + \lambda_3 u^2, \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

where λ_1, λ_2, $\lambda_3 \in \mathbf{R}$.

(1)-(2) is well known nonlinear second-order evolution equations describing various processes in biology[4,5], different methods have been proposed for nonlinear differential equations. But finding particular exact solutions for nonlinear equations of the form (1) remains an important problem. The biological interpretation of (1) is of fundamental importance. In this paper, the solution of the nonlinear differential equation is considered by the differential transformation method. The definition and operation of differential transformations are given. The results of some examples show the advantages of our method for solving nonlinear differential equations.

2 Differential transformation method

In this section, we introduce the basic definition of the differential transformation.

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Definition 1 Suppose $u(x,t)$ is analytic, then its two-dimensional differential transform is expressed as follows

$$U(k,h)=\frac{1}{k!h!}\frac{\partial^{k+h}}{\partial x^k\partial t^h}u(x,t)\Big|_{x=0,t=0}, \tag{3}$$

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h\equiv D^{-1}[U(k,h)], \tag{4}$$

where $u(x,t)$ represents the original function while stands for the transformed function.

Definition 2 Assume $u(x,t)=D^{-1}[U(k,h)]$, $v(x,t)=D^{-1}[V(k,h)]$ and $\otimes$ denote the convolution, then the fundamental operations of two dimensional DTM are defined as follows

$$D[\alpha u(x,t)+\beta v(x,t)]=\alpha U(k,h)+\beta V(k,h), \tag{5}$$

$$D[u(x,t)v(x,t)]=U(k,h)\otimes V(k,h)=\sum_{a=0}^{k}\sum_{b=0}^{h}U(a,h-b)V(k-a,b), \tag{6}$$

$$D[\exp(rx)]=\frac{r^k}{k!},\quad D[x^mt^n]=\delta(k-m,h-n), \tag{7}$$

$$D\Big[\frac{\partial^{r+s}}{\partial x^r\partial t^s}u(x,t)\Big]=(k+1)(k+2)\cdots(k+r)(h+1)(h+2)\cdots(h+s)\\ U(K+r,h+s). \tag{8}$$

Taking differential transforms of (1), we can obtain

$$\begin{aligned}(h+1)U(k,h+1)\;=\;&(k+2)(k+1)U(k+2,h)\\&+\lambda_1\sum_{a=0}^{k}\sum_{b=0}^{h}(k+1-a)U(k+1-a,b)U(a,h-b)\\&+\lambda_2U(k,h)+\lambda_3\sum_{a=0}^{k}\sum_{b=0}^{h}U(a,h-b)U(k-a,b),\end{aligned} \tag{9}$$

From (2), we have

$$u(x,0)=\sum_{k=0}^{\infty}U(k,0)x^k=u_0(x). \tag{10}$$

Substituting all these into (4), we can obtain $u(x,t)$.

3 Numerical illustrations

To illustrate the techniques discussed in the previous section, we give the following examples to validate efficiency of the proposed method.

Example 1 Solve the following Burgers equation ($\lambda_1=-1,\ \lambda_2=\lambda_3=0$ in (1)),

$$u_t-uu_x=u_{xx},\quad 0<x<1,\quad 0<t<1, \tag{11}$$

with the initial value

$$u(x,0) = 1 - x. \tag{12}$$

Taking differential transform of (11), we have

$$\begin{aligned}(h+1)U(k,h+1) &= (k+2)(k+1)U(k+2,h) \\ &\quad + \sum_{a=0}^{k}\sum_{b=0}^{h}(k+1-a)U(k+1-a,b)U(a,h-b),\end{aligned} \tag{13}$$

From (12), we have

$$\sum_{k=0}^{\infty} U(k,0)x^k = 1 - x. \tag{14}$$

The corresponding spectra are

$$U(k,0) = \begin{cases} 1, & k=0, \\ -1, & k=1, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Substituting (15) into (13), we have

$$U(k,1) = \begin{cases} -1, & k=0 \\ 1, & k=1, \\ 0, & \text{otherwise,} \end{cases} \quad U(k,2) = \begin{cases} 1, & k=0, \\ -1, & k=1, \\ 0, & \text{otherwise,} \end{cases} \quad U(k,3) = \begin{cases} -1, & k=0, \\ 1, & k=1, \\ 0, & \text{otherwise,} \end{cases} \quad \cdots.$$

We can calculate $U(k,h)$. Substituting all $U(k,h)$ into (4), we obtain

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = \frac{1-x}{1+t}. \tag{16}$$

Exemple 2 We consider the following heat equation ($\lambda_1 = \lambda_2 = \lambda_3 = 0$ in (1)),

$$u_t = u_{xx} + u, \quad 0 < x < 1, \quad 0 < t < 1, \tag{17}$$

with initial condition

$$u(x,0) = x. \tag{18}$$

Taking the differential transform of (17), we have

$$(h+1)U(k,h+1) = (k+2)(k+1)U(k+2,h) + U(k,h), \tag{19}$$

From (18), we have

$$\sum_{k=0}^{\infty} U(k,0)x^k = x, \tag{20}$$

The corresponding spectra is

$$U(k,0) = \begin{cases} 1, & k=1, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Substituting (21) into (19), we have

$$U(k,1)=\begin{cases}1, & k=1,\\ 0, & \text{otherwise},\end{cases}\quad U(k,2)=\begin{cases}\frac{1}{2}, & k=1,\\ 0, & \text{otherwise},\end{cases}\quad U(k,3)=\begin{cases}\frac{1}{2\cdot 3}1, & k=1,\\ 0, & \text{otherwise},\end{cases}\quad \cdots.$$

We can calculate $U(k,h)$. Substituting all $U(k,h)$ into (4), we obtain

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h = x\Big[1+t+\frac{t^2}{2}+\frac{t^3}{2\cdot 3}+\cdots\Big]=xe'.$$

4 Conclusion

This work shows that the reliability and efficiency of the proposed method and tell us the new method can be an alternative way to solve the linear and nonlinear differential equations.

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非线性微分方程的微分变换方法

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摘 要: 我们用微分变换方法来解非线性微分方程，首先给出微分变换方法的定义和算子，通过求解非线性微分方程的几个实例来验证这个方法的准确性和有效性。

关键词: 非线性微分方程；微分变换方法；初始值问题